Analysis 1, Summer 2023

List 6

Higher derivatives, concavity, local extremes

137. (a) Calculate the derivative of $5x^2 - 3\sin(x)$.
  (b) Calculate the derivative of $10x - 3\cos(x)$.
  (c) Calculate the derivative of $10 + 3\sin(x)$.
  (d) Calculate the derivative of $3\cos(x)$.
  (e) Calculate the derivative of $-3\sin(x)$.

The **second derivative** of a function is the derivative of its derivative. The second derivative of $y = f(x)$ with respect to x can be written as any of

$$f''(x), \qquad f'', \qquad (f')', \qquad f^{(2)}, \qquad y'', \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right], \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

We say f is **twice-differentiable** if f'' exists on the entire domain of f.
Higher derivatives (third, fourth, etc.) are defined and written similarly.

A twice-differentiable function $f(x)$ is **concave up** at $x = a$ if $f''(a) > 0$.
A twice-differentiable function $f(x)$ is **concave down** at $x = a$ if $f''(a) < 0$.

An **inflection point** is a point where the concavity of a function changes.

138. Compute the following second derivatives:
  (a) $f''(x)$ for $f(x) = x$
  (b) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = x^3 + x^8$
  (c) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ for $y = 8x - 4$
  (d) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(5x^2 - 7x + 28)$
  (e) $f''(x)$ for $f(x) = -2x^8 + x^6 - x^3$
  (f) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = ax^2 + bx + c$

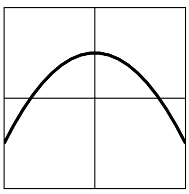
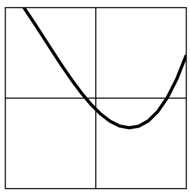
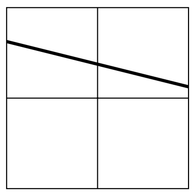
139. Find $f'''(x) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = f^{(3)}(x)$ (the third derivative) for $f(x) = x^7$.

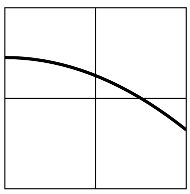
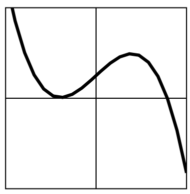
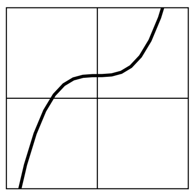
140. Give $f^{(5)}(x) = \frac{\mathrm{d}^5 f}{\mathrm{d}x^5}$ (the fifth derivative) for $f(x) = 5x^2 - 3\sin(x)$.

141. (a) Is the function $3x^2 + 8\cos(x)$ concave up or concave down at $x = 0$?
  (b) Is the function $3x^2 + 5\cos(x)$ concave up or concave down at $x = 0$?

142. On what interval(s) is $54x^2 - x^4$ concave up?

143. For each of the following functions, is $f''(0)$ is positive, zero, or negative?

(a) 
(b) 
(c) 

(d) 
(e) 
(f) 

144. For $f(x) = x^3 - x^2 - x$,

(a) At what x value(s) does $f(x)$ change sign? That is, list values r where either $f(x) < 0$ when x is slightly less than r and $f(x) > 0$ when x is slightly more than r, or $f(x) > 0$ when x is slightly less than r and $f(x) < 0$ when x is slightly more than r.

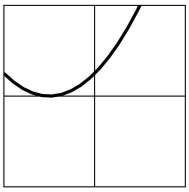
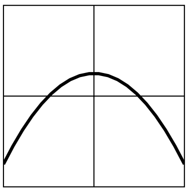
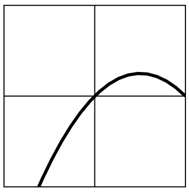
(b) At what x value(s) does $f'(x)$ change sign?

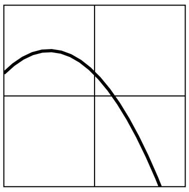
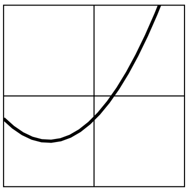
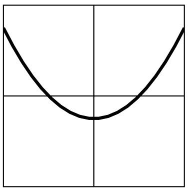
(c) At what x value(s) does $f''(x)$ change sign?

(d) List all inflection points of $x^3 - x^2 - x$.

☆145. Give an example of a function with one local maximum and two local minimums but no inflection points.

146. Which graph below has $f'(0) = 1$ and $f''(0) = -1$?

(A)  (B)  (C) 

(D)  (E)  (F) 

For a twice-differentiable function $f(x)$ with a critical point at $x = c$, ...

The Second Derivative Test:
- If $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f''(c) < 0$ then f has a local maximum at $x = c$.
- If $f''(c) = 0$ the test is inconclusive.

The First Derivative Test:
- If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then f has a local minimum at $x = c$.
- If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then f has a local maxium at $x = c$.
- If $f'(x)$ has the same sign on both sides of $x = c$ then $x = c$ is neither a local minimum nor a local maximum.

147. Find all critical points of

$$4x^3 + 21x^2 - 24x + 19$$

and classify each as a local minimum, local maximum, or neither.

148. Find and classify[1] the critical points of

$$f(x) = x^4 - 4x^3 - 36x^2 + 18.$$

[1]"Classify the critical points" means to say whether each one is a local minimum, local maximum, or neither.

149. Find the inflection points of the function from Task 148.

☆150. Find and classify the critical points of $f(x) = x(6-x)^{2/3}$.

151. Find and classify the critical points of

$$\tfrac{3}{2}x^4 - 16x^3 + 63x^2 - 108x + 51.$$

152. Label each of following statements as "true" or "false":
   (a) Every critical point of a differentiable function is also a local minimum.
   (b) Every local minimum of a differentiable function is also a critical point.
   (c) Every critical point of a differentiable function is also an inflection point.
   (d) Every inflection point of a differentiable function is also a critical point.

153. A twice-differentiable function $f(x)$ has the following properties:

$$\begin{array}{lll} f(4) = 2 & f'(4) = 18 & f''(4) = 0, \\ f(7) = 19 & f'(7) = 0 & f''(7) = -1. \end{array}$$

Label each of following statements as "true", "false", or "cannot be determined":
   (a) f has a critical point at $x = 4$.
   (b) f has a local maximum at $x = 4$.
   (c) f has an absolute maximum at $x = 4$.
   (d) f has an inflection point at $x = 4$.
   (e) f has a critical point at $x = 7$.
   (f) f has a local maximum at $x = 7$.
   (g) f has an absolute maximum at $x = 7$.
   (h) f has an inflection point at $x = 7$.

☆154. What is the maximum number of inflection points that a function of the form

$$\_\_x^6 + \_\_x^5 + \_\_x^4 + \_\_x^3 + \_\_x^2 + \_\_x + \_\_$$

can have?

Individual functions: $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^p\right] = px^{p-1}$, $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x\right] = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x)\right] = \frac{1}{x}$,
$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x)\right] = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(x)\right] = -\sin(x)$.

Sum Rule: $(f+g)' = f' + g'$ **Product Rule:** $(f \cdot g)' = fg' + f'g$

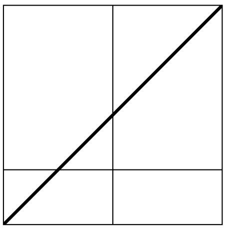
Chain Rule: $(f(g))' = f'(g) \cdot g'$ **Quotient Rule:** $(f/g)' = \dfrac{gf' - fg'}{g^2}$

155. Give an equation for the tangent line to $y = \sin(\pi x)$ at $x = 2$.

156. Find the derivative of $\sin(5^{\cos(2x^3+8)})$.

157. Match the functions (a)-(g) to their *second* derivatives (I)-(VII).

(a)



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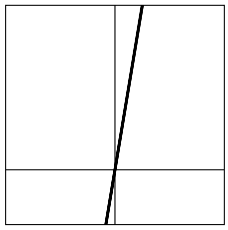

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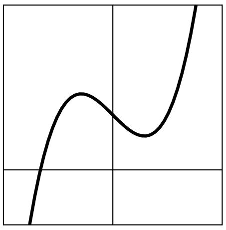

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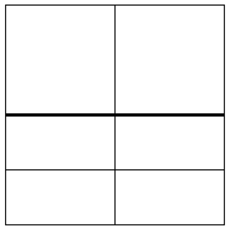

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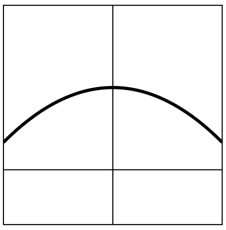

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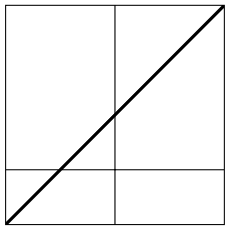

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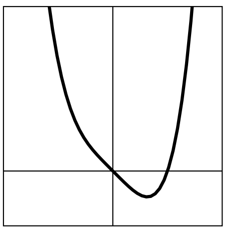

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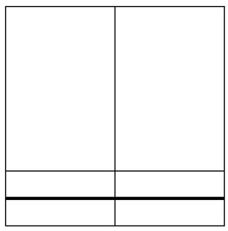
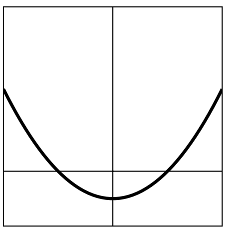
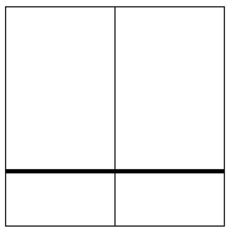


(e)



(V)



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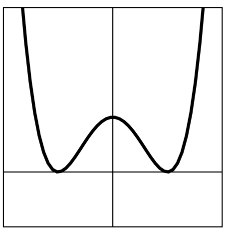
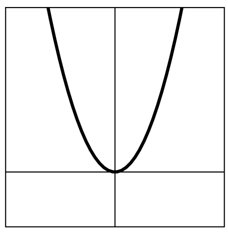
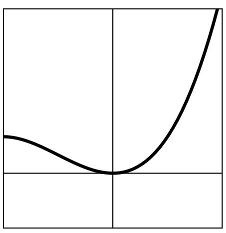


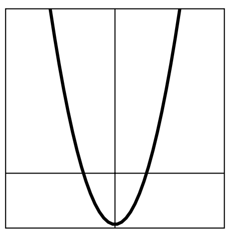
(VI)



(g)



(VII)



158. (a) At $x = 2$, is $\dfrac{x^2}{1+x^3}$ increasing, decreasing, or neither?

(b) At $x = 0$, is $\ln(2 + \sin(x))$ concave up, concave down, or neither?

(c) At $x = \frac{3\pi}{4}$, does $e^x \sin(x)$ have a local minimum, local maximum, or neither?

159. Find the critical points of $e^{(x^2+8x)}$ and classify each as a local min or max.

160. Find the x- and y-value of the absolute minimum of $y = f(x) = \cos(\sqrt{x})$ on the interval $\frac{1}{4}\pi^2 \le x \le 4\pi^2$.

161. (a) Use the Quotient Rule to differentiate $\dfrac{\sin(x)}{x^4}$.

(b) Use the Product Rule to differentiate $x^{-4}\sin(x)$.

(c) Use algebra to compare your answers from parts (a) and (b).

162. Match the functions (a)-(d) with their derivatives (I)-(IV).

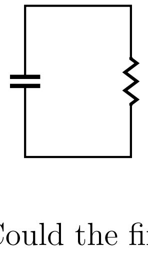
(a) $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$	(I) $\sec(x)\tan(x) = \dfrac{\sin(x)}{(\cos(x))^2}$
(b) $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$	(II) $-(\csc(x))^2 = \dfrac{-1}{(\sin(x))^2}$
(c) $\sec(x) = \dfrac{1}{\cos(x)}$	(III) $(\sec(x))^2 = \dfrac{1}{(\cos(x))^2}$
(d) $\csc(x) = \dfrac{1}{\sin(x)}$	(IV) $-\csc(x)\cot(x) = \dfrac{-\cos(x)}{(\sin(x))^2}$

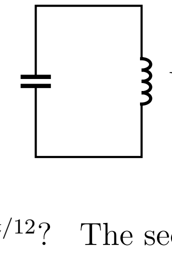
163. Instead of memorizing a formula for $\frac{d}{dx}[a^x]$, you could memorize $\frac{d}{dx}[e^x] = e^x$ and use about facts about logs/exponents, such as

$$4^x = \left(e^{\ln 4}\right)^x = \left(e^{1.386}\right)^x = e^{(1.386\,x)}.$$

Use the Chain Rule to find the derivative of this function. In general, what is the derivative of $e^{g(x)}$?

☆164. Below are two circuits and an equation involving a derivative for each of them. Here R (resistance), C (capacitance), and L (inductance) are all constants, but the voltage $V = V(t)$ is a function of time.

 $V + RC\dfrac{dV}{dt} = 0$

 $V + LC\dfrac{d^2V}{dt^2} = 0$

(a) Could the first circuit have $V(t) = 5e^{-t/12}$? The second circuit?

(b) Could the first circuit have $V(t) = 5(1 - e^{-t/12})$? The second circuit?

(c) Could the first circuit have $V(t) = 5\sin(2\pi t)$? The second circuit?

(d) Could the first circuit have $V(t) = 0$? The second circuit?

165. The "information entropy" h of a weighted coin where Heads (Orzel) has probability p_1 and Tails (Reszka) has probability p_2 is

$$-p_1 \ln(p_1) - p_2 \ln(p_2).$$

Labeling $x = p_1$, we have $p_2 = 1 - x$ (because probabilities must add to 1), so the entropy of a coin with $P(\text{Heads}) = x$ is

$$h(x) \quad = \quad -x \ln(x) - (1-x)\ln(1-x).$$

Determine the x-value that gives the maximum of $h(x)$.
(This will be the *most random* kind of coin.)

166. Give the one hundredth derivative of $x\,e^x$, that is, $\dfrac{\mathrm{d}^{100}}{\mathrm{d}x^{100}}\big[x\,e^x\big]$.

167. Find the following derivatives (note (p)-(ż) require the Chain Rule).

(a) $f'(x)$ for $f(x) = x^9 \sin(x)$

(ą) $\frac{\mathrm{d}}{\mathrm{d}x}\big(x^9 \sin(x)\big)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\big(10^x + \log_{10}(x)\big)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\big(10^x \cdot \log_{10}(x)\big)$

(ć) $\frac{\mathrm{d}}{\mathrm{d}x}\big(\sqrt{x}\sin(x)\big)$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\big(x^9\, e^x \sin(x)\big)$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}(4x^3 + x\sin x)$

(ę) $\frac{\mathrm{d}}{\mathrm{d}t}(4t^3 + t\sin t)$

(f) $\frac{\mathrm{d}}{\mathrm{d}t}\ \sin(t)\cos(t)$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\dfrac{\cos(x)}{5x^3 - 12}$

(h) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\dfrac{5x^3 - 12}{\cos(x)}$

(i) $\dfrac{\mathrm{d}}{\mathrm{d}t}\,\dfrac{t^7 + t^2}{e^t}$

(j) $\frac{\mathrm{d}}{\mathrm{d}x}(5x - 7)^2$

(k) $\frac{\mathrm{d}}{\mathrm{d}t}\ e^t \cos(t)$

(l) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(t\sin(t) + \dfrac{e^t}{t^2+1}\right)$

(ł) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\dfrac{\sin(t)}{t\,e^t}$

(m) $\frac{\mathrm{d}}{\mathrm{d}t}\ t^{5/2}\sin(t)$

(n) $\frac{\mathrm{d}}{\mathrm{d}x}\ 2^{15}$

(ń) $\frac{\mathrm{d}}{\mathrm{d}x}\ x^{15}$

(o) $\frac{\mathrm{d}}{\mathrm{d}u}\ u^{15}$

(ó) $\frac{\mathrm{d}}{\mathrm{d}x}\ u^{15}$ if u is a constant

(p) $\frac{\mathrm{d}}{\mathrm{d}x}\ u^{15}$ if u is a fn. of x

(q) $\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x))^{15}$

(r) $\frac{\mathrm{d}}{\mathrm{d}x}\ \ln(\cos(x))$

(s) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\ln(\cos(x))}$

(ś) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ e^{\sqrt{\ln(\cos(x))}}$

(t) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ e^{\sqrt{\ln(\cos(x^6))}}$

(u) $\frac{\mathrm{d}}{\mathrm{d}t}\ 5\sin(2t+1)$

(v) $\frac{\mathrm{d}}{\mathrm{d}t}\ A\sin(\omega t + \phi)$ if A, ω, t are constants

(w) $\frac{\mathrm{d}}{\mathrm{d}x}(7x^2 + \sin(x))^2$

(x) $\frac{\mathrm{d}}{\mathrm{d}x}(\log_3(x))^2$

(y) $\frac{\mathrm{d}}{\mathrm{d}t}\ \tan(t^3 + 8t^2 + 2t + 18)$

(z) $\frac{\mathrm{d}}{\mathrm{d}x}\ \cos(x^3 e^x)$

(ź) $\frac{\mathrm{d}}{\mathrm{d}x}\ x^3\cos(9x)$

(ż) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\dfrac{x^3\cos(x)}{e^{\sin(x)}}$